UNIVERSITY OF LEICESTER

School of Engineering

FEM and Elasticity Theory

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EG3111 – Finite Element Analysis and Design

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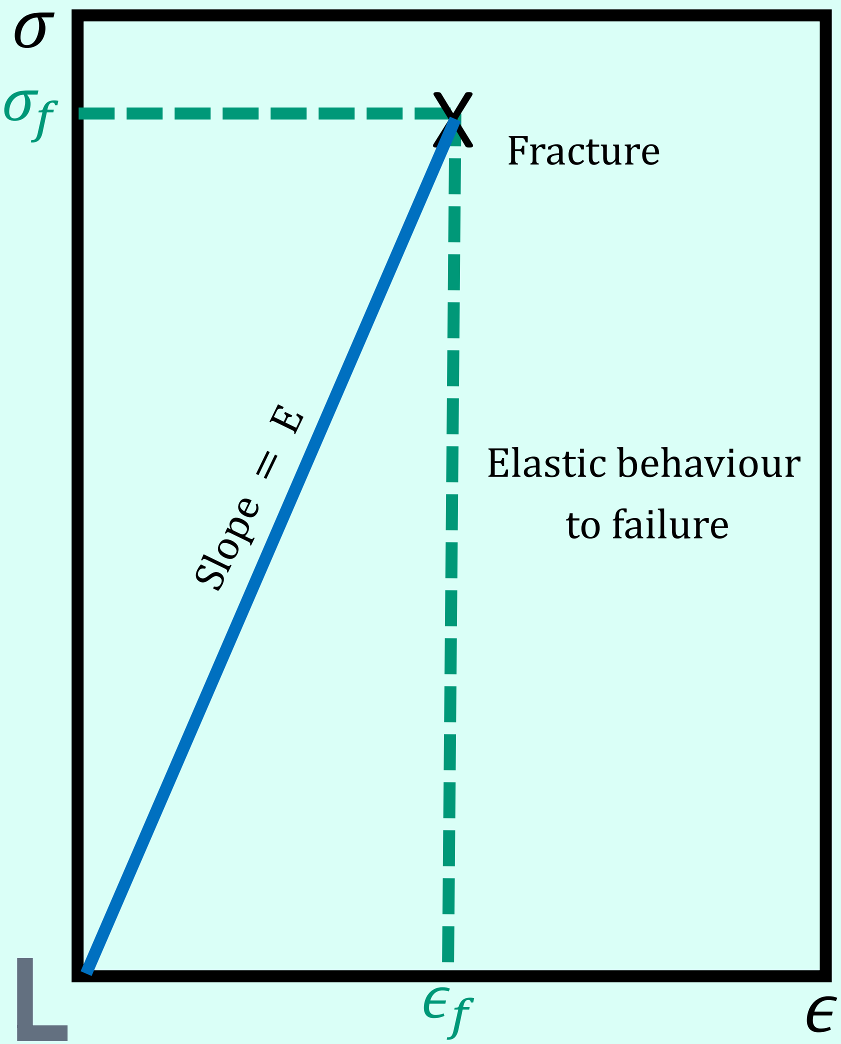
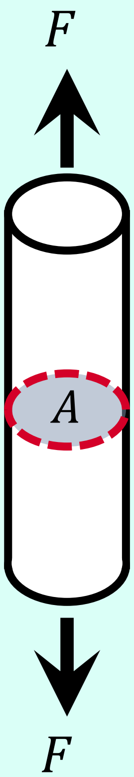
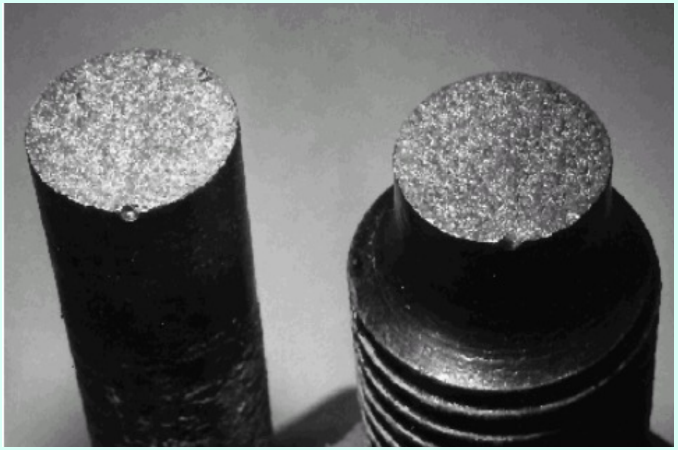
2a. Recap on Deformation – Brittle Materials from EG2111



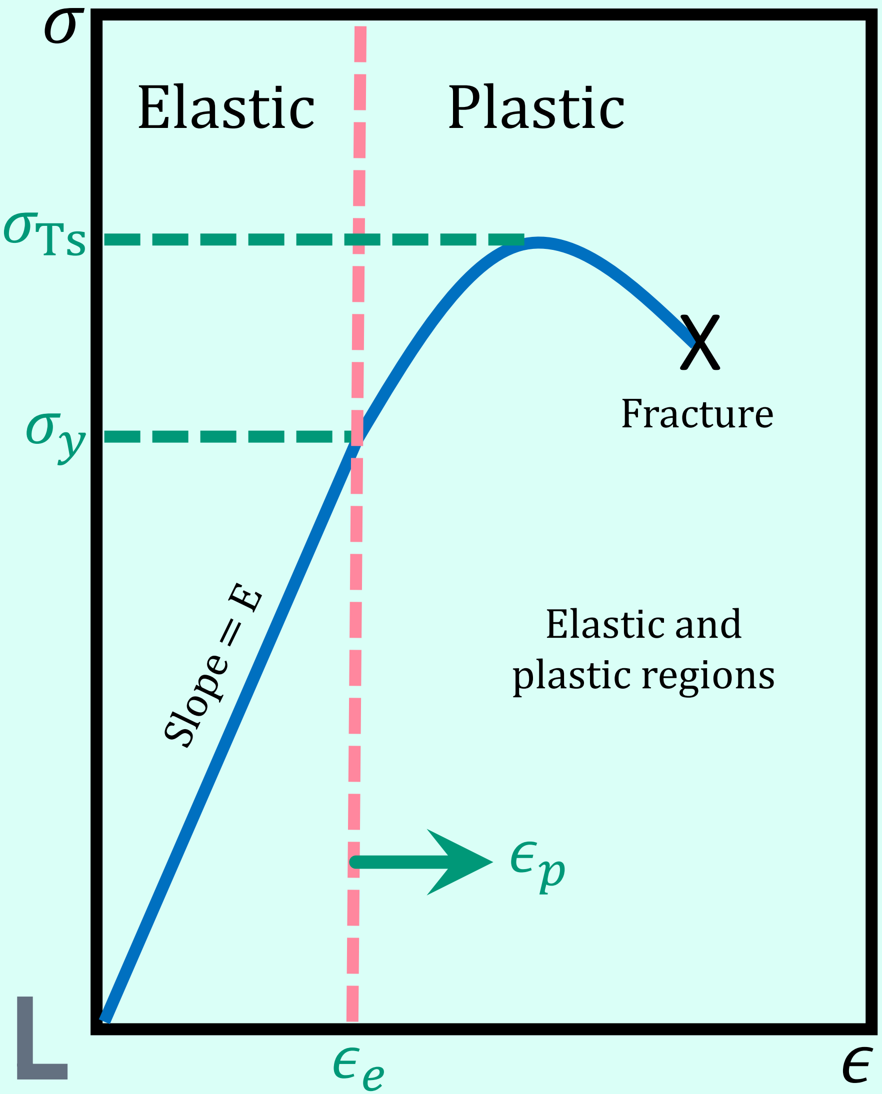


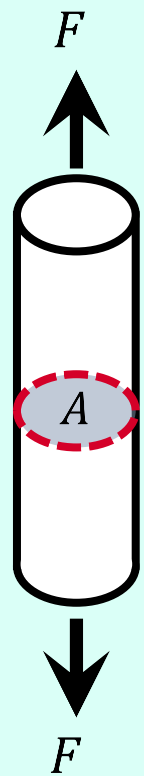


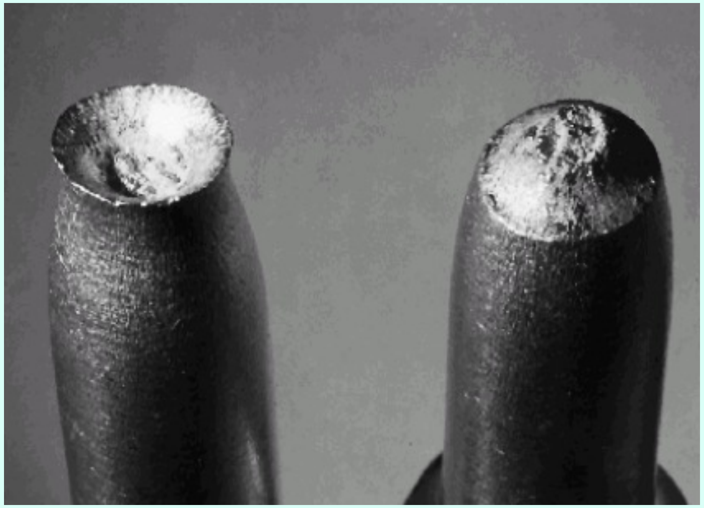
Brittle fracture in mild steel
http://people.virginia.edu/~lz2n/mse209/Chapter8.pdf

2a. Recap on Deformation – Ductile Materials

Total strain, $\boldsymbol{\epsilon} = \epsilon_e + \epsilon_p = \frac{\sigma_y}{E} + \epsilon_p$

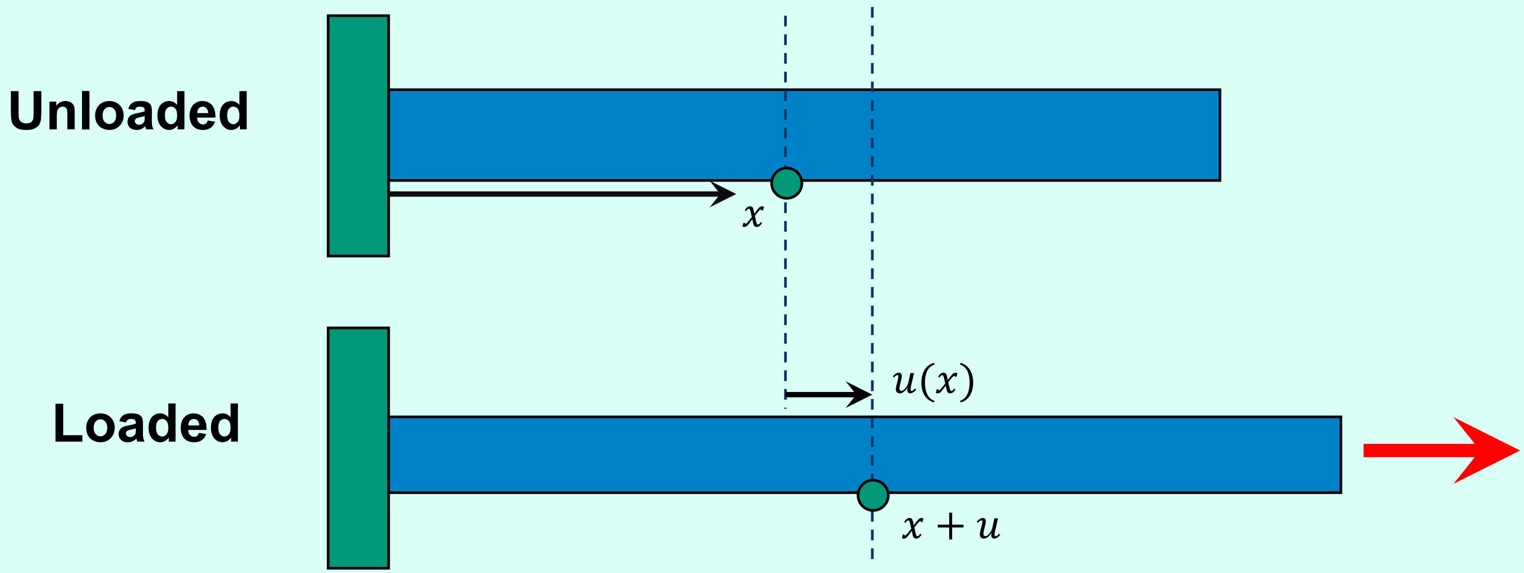






Cup-and-cone fracture in Al
http://people.virginia.edu/~lz2n/mse209/Chapter8.pdf

2a. PDE for 1D elasticity



When a load is applied, a point at position x moves to position $x + u$.

The solution to this 1D elasticity problem is the <u>displacement field $\mathbf{u}(\mathbf{x})$.</u>

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2a. PDE for 1D elasticity

Three conditions to satisfy:

- **Compatibility of strains**
- **Force balance**
- **Constitutive law**

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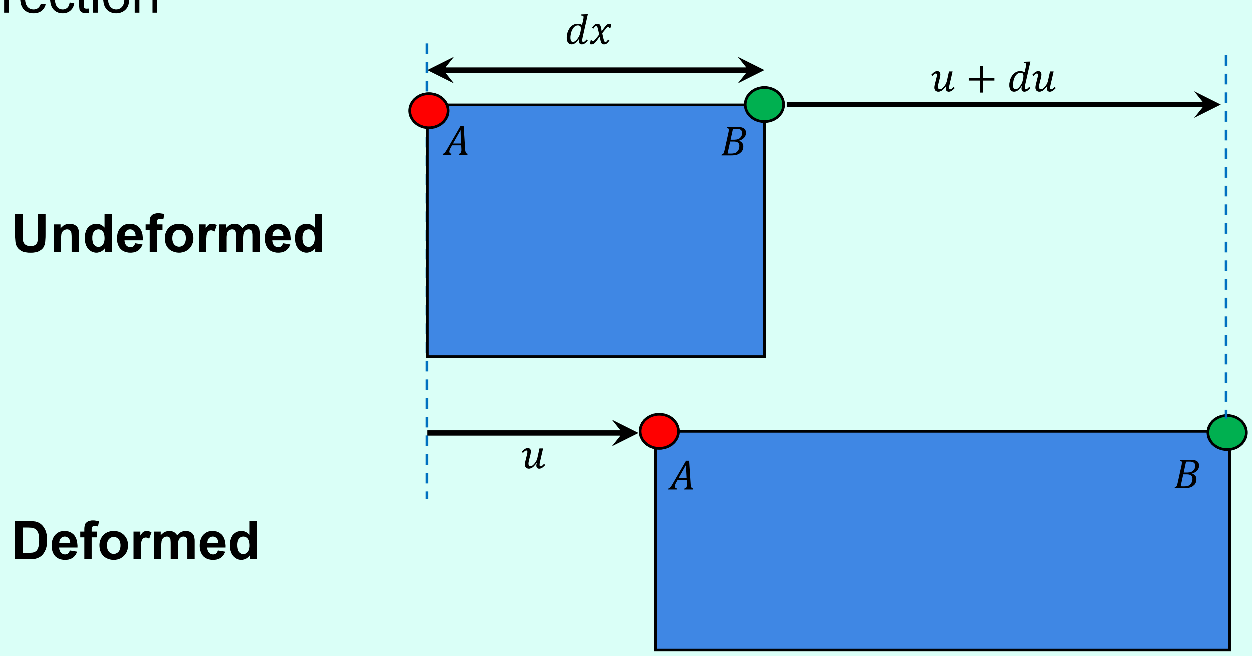
2a. Compatibility of strain (recap from EG2111)

In x-direction



- **Normal** strain in x-direction ϵ_x is change in length over original length

$$\epsilon_x = \frac{du}{dx}$$

2a. PDE for 1D elasticity

Three conditions to satisfy:

- **Compatibility of strains**

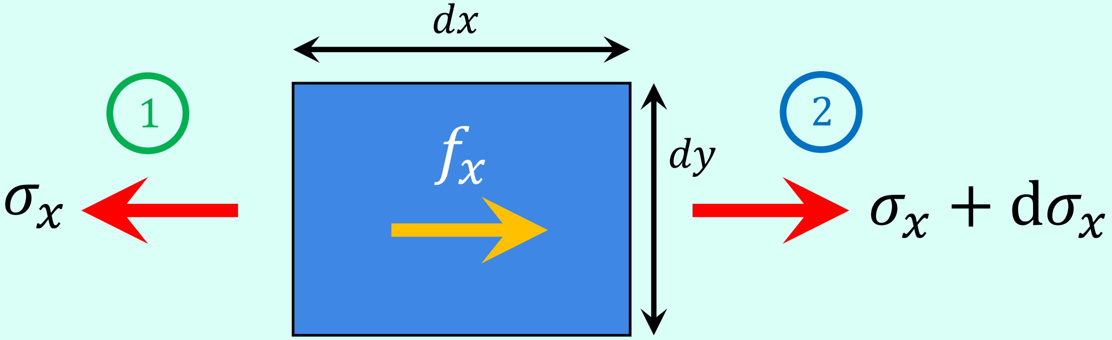
$$\epsilon_x = \frac{du}{dx}$$

- **Force balance**

- **Constitutive law**

2a. Force balance

Consider an infinitesimal element subject to an internal body force per unit volume f_x, e.g. an inertial force such as gravity.



Force balance (1) (2)

$$\sigma_x dy - (\sigma_x + d\sigma_x)dy - f_x dx.dy = 0$$

$$\sigma_x dy - (\sigma_x + d\sigma_x)dy - f_x dx.dy = 0 \implies -d\sigma_x - f_x dx = 0$$

Divide by dx and write as

$$\boxed{\frac{d\sigma_x}{dx} + f_x = 0}$$

2a. PDE for 1D elasticity

Three conditions to satisfy:

- **Compatibility of strains**

$$\epsilon_x = \frac{du}{dx}$$

- **Force balance**

$$\frac{d\sigma_x}{dx} + f_x = 0$$

- **Constitutive law**

2a. Constitutive law

Young's modulus, E (Pa or N/m^2)

$$\sigma_x = E\epsilon_x$$

$$\frac{d\sigma_x}{dx} = E\,\frac{d\epsilon_x}{dx}$$

Compatibility of strains:

$$\epsilon_x = \frac{du}{dx}$$

$$\frac{d\sigma_x}{dx} = E\,\frac{d}{dx}\left(\frac{du}{dx}\right) = E\,\frac{d^2u}{dx^2}$$

Force balance:

$$\frac{d\sigma_x}{dx} + f_x = 0$$

$$\boxed{E\,\frac{d^2u}{dx^2} + f_x = 0}$$

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2a. PDE for 1D elasticity

Three conditions to satisfy:

- **Compatibility of strains**

$$\epsilon_x = \frac{du}{dx}$$

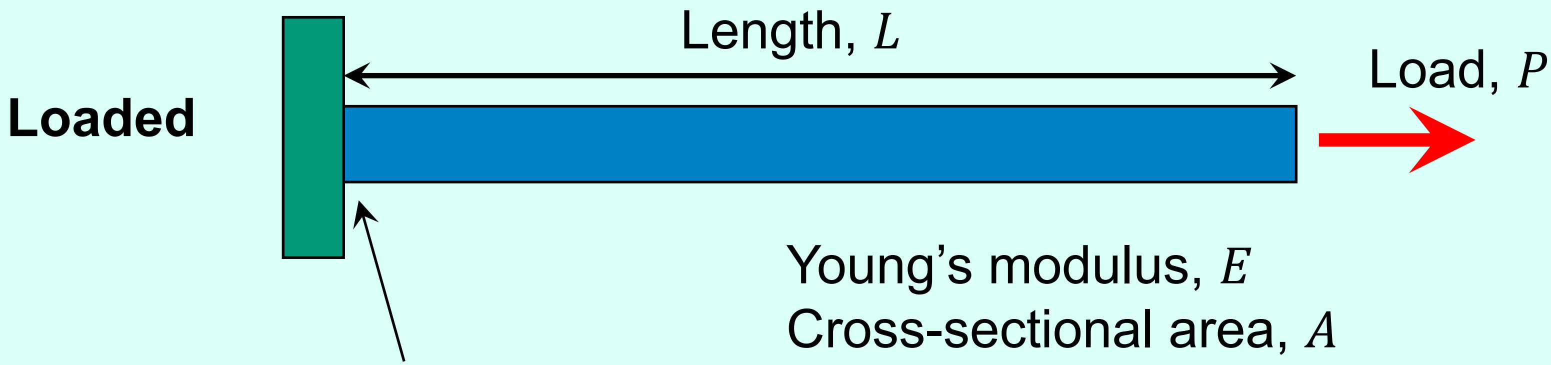
- **Force balance**

$$\frac{d\sigma_x}{dx} + f_x = 0$$

- **Constitutive law**

$$E\frac{d^2u}{dx^2} + f_x = 0$$

2a. (i) Simple 1D extension



End condition $u(0) = 0$

$$E\frac{d^2u}{dx^2} + f_x = 0$$

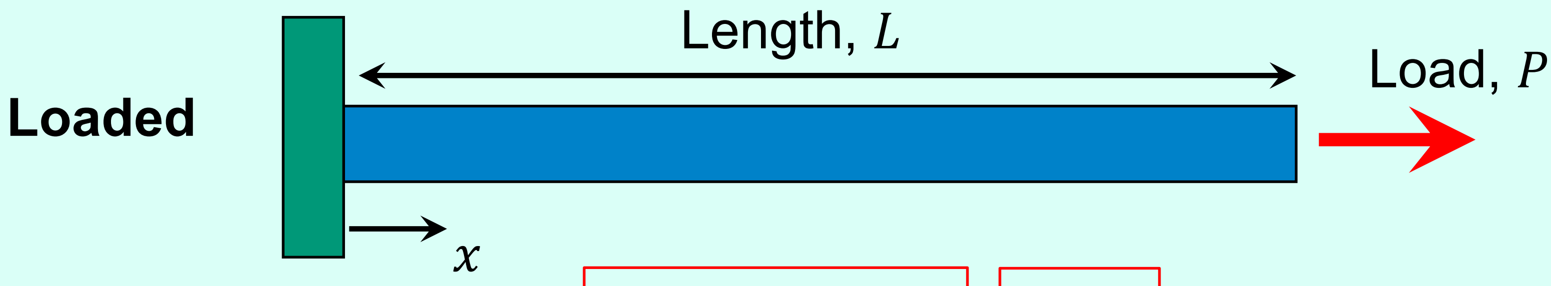
No body force: $f_x = 0$

$$E\frac{d^2u}{dx^2} = 0 \quad \Longrightarrow \quad \frac{d^2u}{dx^2} = 0$$

Integrate:

$$\frac{du}{dx} = b \quad \Longrightarrow \quad u(x) = a + bx$$

2a. (i) Simple 1D extension



$$u(x) = a + bx \qquad Eq\ 1$$

End condition “bar is fixed at $x = 0$

$$u(0) = 0$$

Substitute in $Eq\ 1$:

$$u(0) = a = 0$$

2a. (i) Simple 1D extension

A linear shape function is adequate in this case as the exact solution is linear

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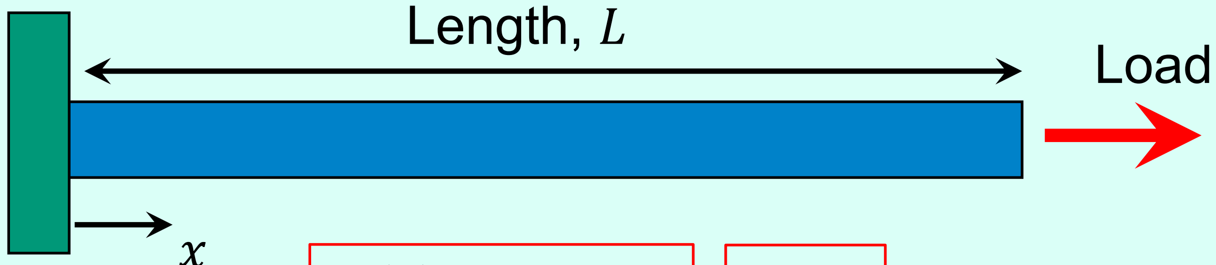


$$u(x) = a + bx \qquad Eq\ 1$$

Differentiate $Eq\ 1$

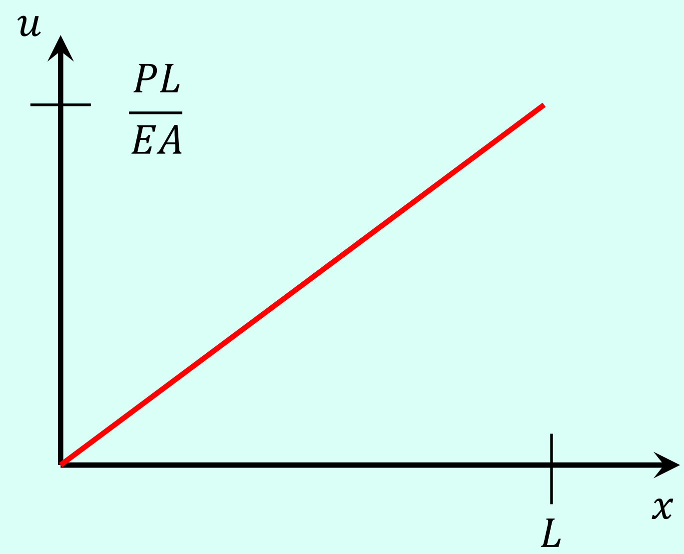
$$\frac{du}{dx} = b$$

At other end:

Stress, $\sigma_x = \frac{P}{A} = E\epsilon_x = E\frac{du}{dx} = Eb$

$$\frac{P}{A} = Eb \quad \Longrightarrow \quad b = \frac{P}{EA}$$

Substitute in $Eq\ 1$: $\Longrightarrow$ $u(x) = \frac{P}{EA}x$



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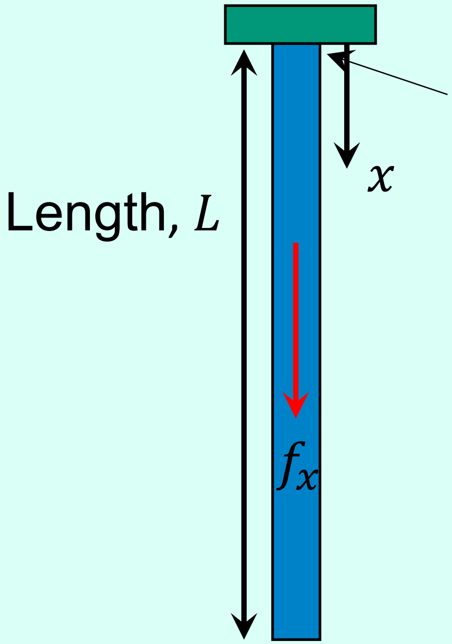
2a. (ii) Self-weight (gravity)



End condition $u(0) = 0$

$$E \frac{d^2u}{dx^2} + f_x = 0 \qquad Eq\ 2$$

Body force f_x is due to gravity

Total force $= mg$

Total force per unit volume $= \frac{mg}{V}$

Body force: $f_x = \rho g$

Density, $\rho = \frac{mass}{Volume} = \frac{m}{V}$

Substitute f_x in $Eq\ 2$: $\Longrightarrow \quad E \frac{d^2u}{dx^2} + \rho g = 0$

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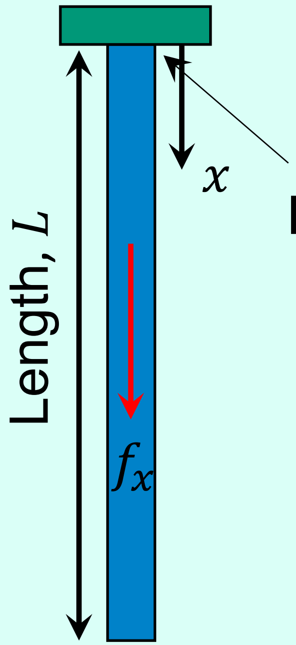
2a. (ii) Self-weight (gravity)



End condition
$u(0) = a = 0$

$$E\frac{d^2u}{dx^2} + \rho g = 0 \quad \Longrightarrow \quad \frac{d^2u}{dx^2} = -\frac{\rho g}{E}$$

Integrate twice:

$$\frac{du}{dx} = b - \frac{\rho g x}{E}$$

$$\boxed{u(x) = a + bx - \frac{\rho g}{2E}x^2}$$

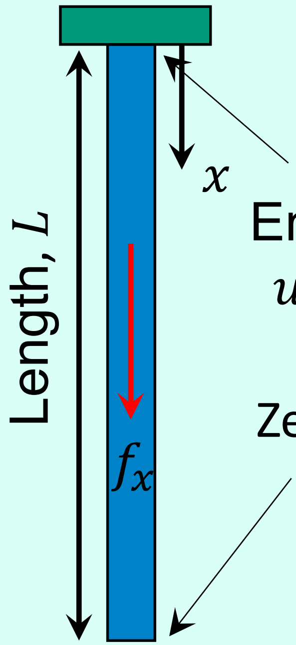
Two End Conditions:

Top end:

$$u(0) = 0 \Longrightarrow \quad a = 0$$

2a. (ii) Self-weight (gravity)

$$u(x) = a + bx - \frac{\rho g}{2E} x^2 \qquad Eq\ 3$$



End condition
$u(0) = a = 0$

Zero stress at $x = L$

Bottom end:

$$\sigma_x = \frac{P}{A} = 0 = E\epsilon_x = E\frac{du}{dx} = 0$$

$$\left.\frac{du}{dx}\right|_{x=L} = 0$$

Differentiate $Eq\ 3$

$$\left.\frac{du}{dx}\right|_{x=L} = 0 = b - \left.\frac{\rho g x}{E}\right|_{x=L}$$

$$b = \frac{\rho g L}{E}$$

$$\sigma_x = Eb - \rho g x = Eb - \rho g L = 0$$

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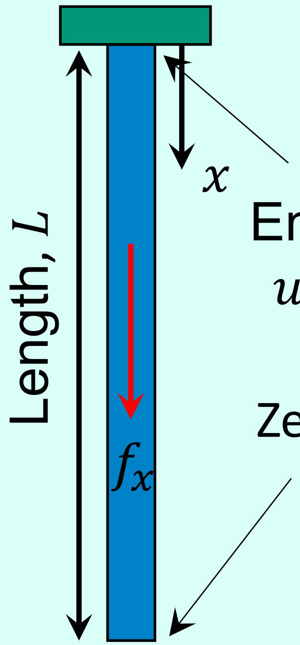
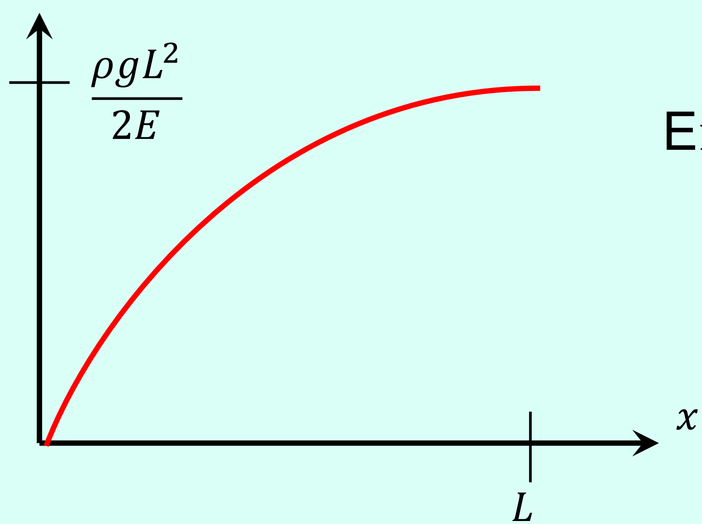
2a. (ii) Self-weight (gravity)



End condition

$$u(0) = a = 0$$

Zero stress at $x = L$



$$a = 0$$

$$b = \frac{\rho g L}{E}$$

Substitute a and b in:

$$u(x) = a + bx - \frac{\rho g}{2E} x^2$$

$$u(x) = \frac{\rho g L}{E}\left(x - \frac{x^2}{2L}\right)$$

End dispacement:

$$u(L) = \frac{\rho g L^2}{2E}$$