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School of Engineering

Bar Elements

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EG3111 – Finite Element Analysis and Design

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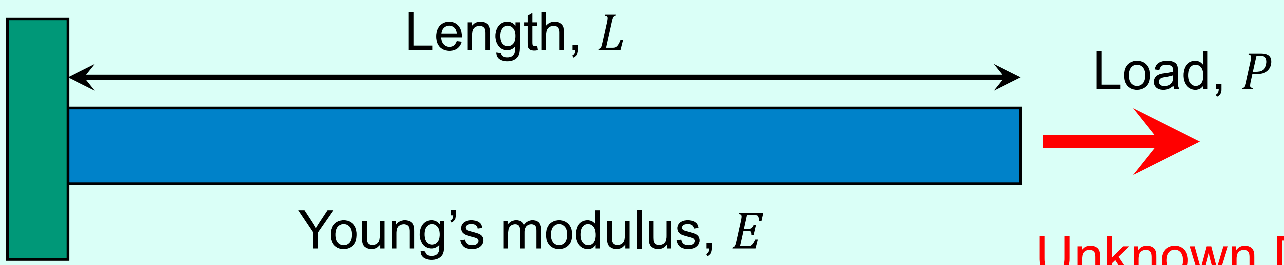
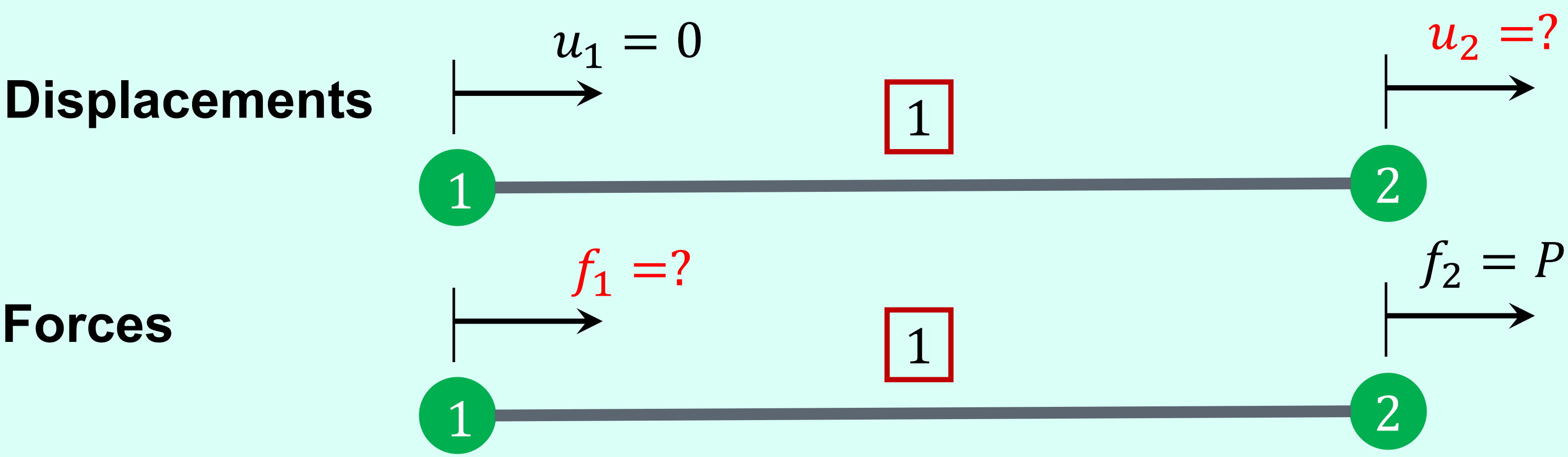
3b. Bar Element Example 1



Unknown DOF in red.



In general, either the displacement OR the force at a node is known, but never NEITHER or BOTH.

$$\underline{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ P \end{bmatrix}$$

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3b. Bar Element Example 1

All that is needed is the elemental stiffness matrix.

For element (1)

$$[k^{(1)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \underline{d}^{(1)} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

As only one element the elemental matrix is the same as the global matrix so no “assembly” required.

$$[K] = [k^{(1)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{[K].\underline{d} = \underline{f}} \qquad \Rightarrow \qquad \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ P \end{bmatrix}$$

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3b. Bar Element Example 1

Two equations for 2 unknowns (u_2 and f_1)

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ P \end{bmatrix}$$

$$\frac{EA}{L}(1 \times 0 - 1 \times u_2) = f_1 \qquad \boxed{Eq\ 1}$$

$$\frac{EA}{L}(-1 \times 0 + 1 \times u_2) = P \qquad \boxed{Eq\ 2}$$

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3b. Bar Element Example 1

Second equation gives

$$\frac{EA}{L}(-1 \times 0 + 1 \times u_2) = P \quad \overset{\textit{Partition}}{\Longrightarrow} \quad \frac{EA}{L}u_2 = P$$

$$u_2 = \frac{PL}{EA}$$

Same as previous solution as same linear shape function

First equation gives

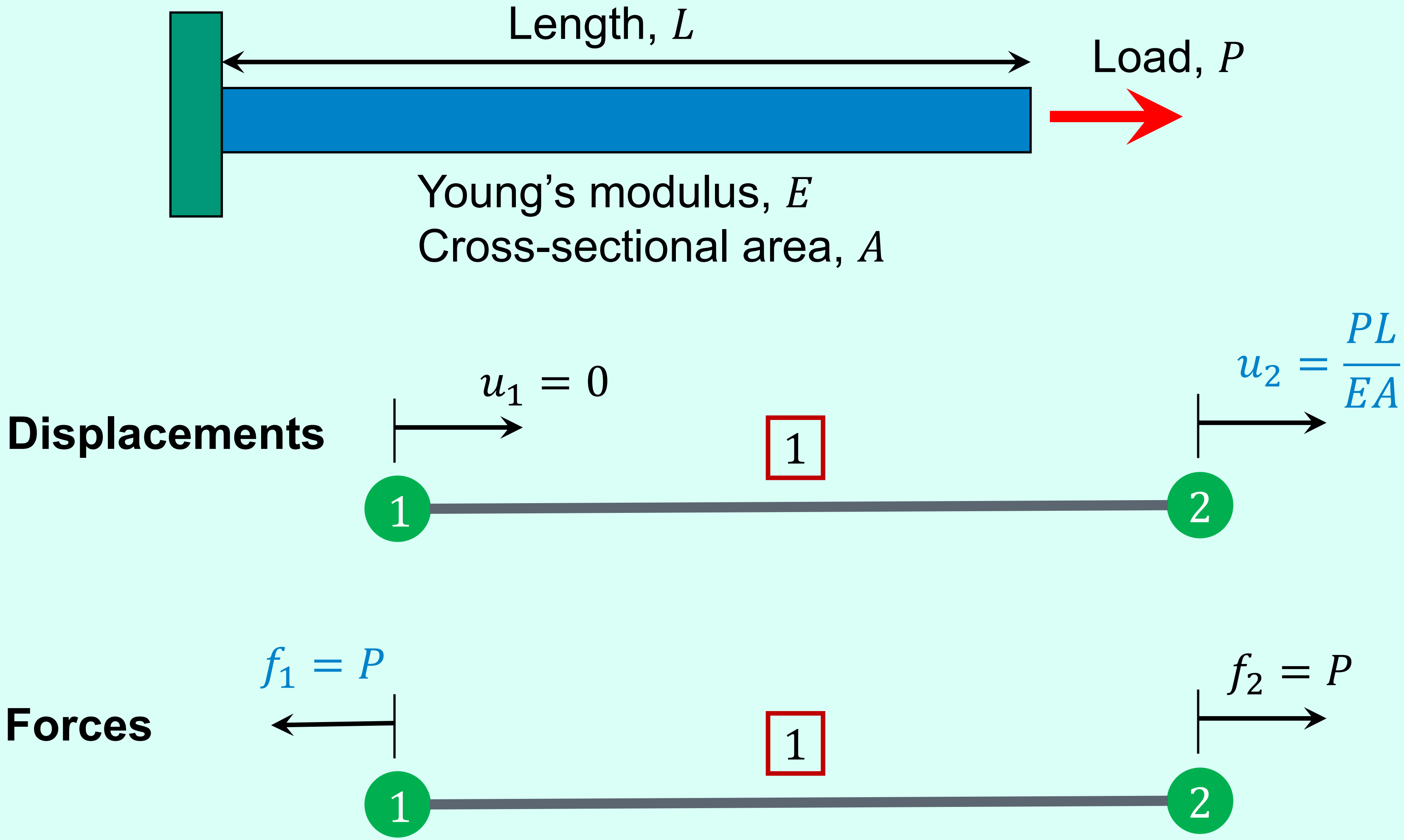
$$\frac{EA}{L}(1 \times 0 - 1 \times u_2) = f_1 \quad \Longrightarrow \quad -\frac{EA}{L}u_2 = f_1$$

Substitute u_2 in $Eq\ 1$

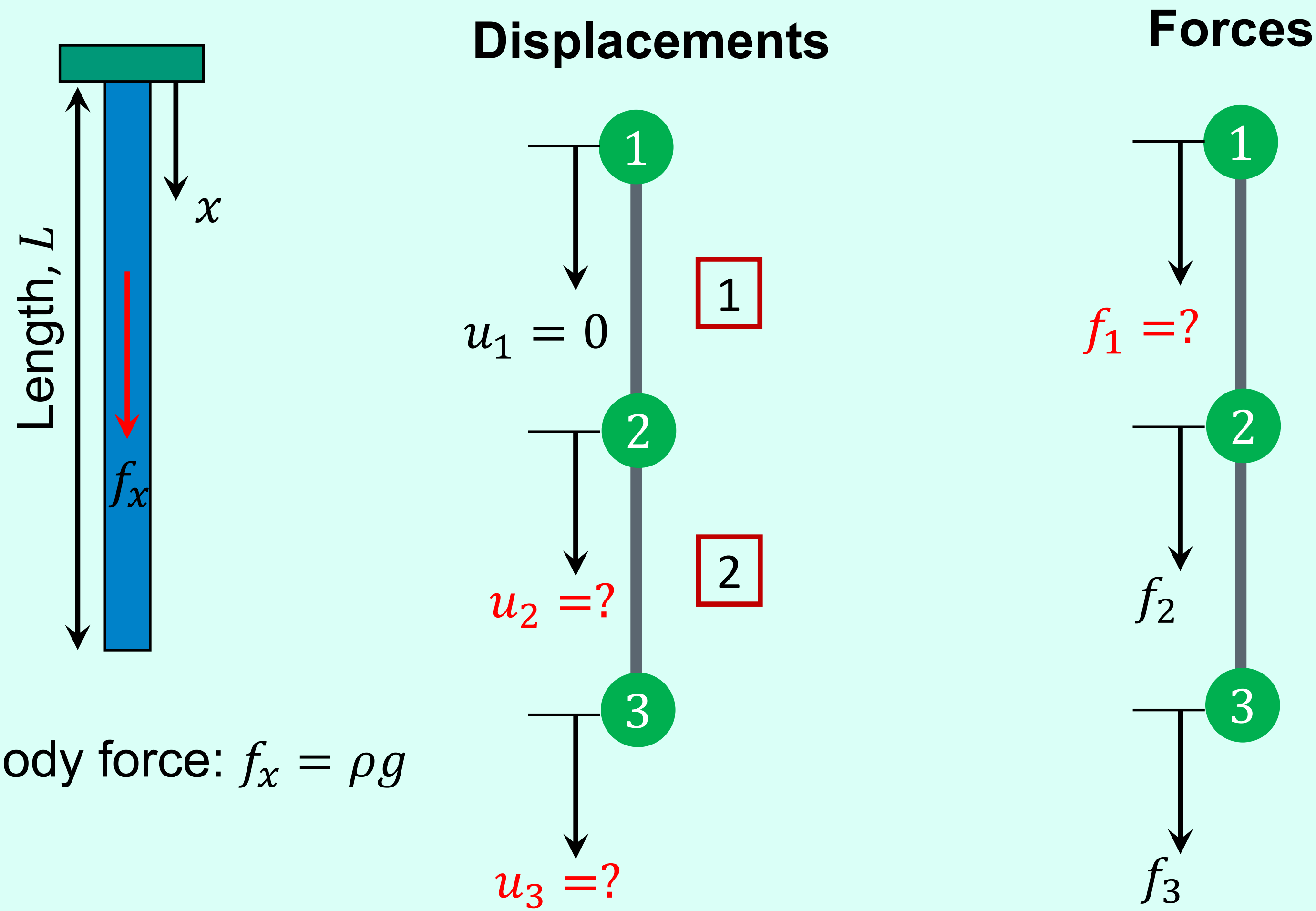
$$f_1 = -P$$

This is the reaction force provided by the wall to keep the displacement $u_1 = 0$.

3b. Bar Element Example 1



3b. Bar Element Example 2 : Self weight



Body force: $f_x = \rho g$

From section 2b, we know exact solution is a quadratic. Here we use two linear bar elements to get an approximate solution.

3b. Bar Element Example 2 : Self weight

Forces due to distributed load

$$\Omega^e = -\int_{V^e} f_x u(x)\,dx = -AL^e \int_0^1 \rho g.\underline{n^{e^T}}(\xi)d\xi.\underline{d^e} = -\underline{f^{e^T}}.\underline{d^e}$$

So

$$\underline{f^e} = AL^e\rho g \int_0^1 \underline{n^e}(\xi)d\xi = AL^e\rho g \int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} d\xi = AL^e\rho g \begin{bmatrix} \xi - \frac{1}{2}\xi^2 \\ \frac{1}{2}\xi^2 \end{bmatrix}_0^1$$

$$\underline{f^e} = AL^e\rho g \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{AL^e\rho g}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Where $L^e = \frac{L}{2}$ Each element is half the total length L

3b. Bar Element Example 2 : Self weight

Global matrices

$$\underline{d} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Elemental stiffness matrices

Element (1)

$$\left[k^{(1)}\right] = \frac{EA}{\frac{1}{2}L} \begin{array}{cc} \begin{array}{cc} u_1 & u_2 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \end{array} \end{array}$$

$$\underline{d}^{(1)} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Applied load due to self weight

Force due to unknown reaction at node 1

$$\underline{f}^{(1)} = \frac{A\frac{L}{2}\rho g}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{AL\rho g}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$$

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3b. Bar Element Example 2 : Self weight

Element (2)

$$[k^{(2)}] = \frac{EA}{\frac{1}{2}L}\begin{array}{cc} & \begin{array}{cc} u_2 & u_3 \end{array} \\ & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} u_2 \\ u_3 \end{array} \end{array}$$

$$\underline{d^{(2)}} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$\underline{f^{(2)}} = \frac{AL\rho g}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} = \frac{AL\rho g}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a = \frac{AL\rho g}{4}$$

3b. Bar Element Example 2 : Self weight

Assembly of elemental force matrices

$$\underline{f} = \begin{bmatrix} a + f_1 \\ a + a \\ a \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}$$

Element (1)

$$\underline{f^{(1)}} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_1 \\ 0 \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \end{array}$$

Element (2)

$$\underline{f^{(2)}} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{array}{c} u_2 \\ u_3 \end{array}$$

3b. Bar Element Example 2 : Self weight

Global stiffness matrix

$$[K] = \frac{2EA}{L}\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array}$$

Element (1)

$$[k^{(1)}] = \frac{2EA}{L}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array}$$

Element (2)

$$[k^{(2)}] = \frac{2EA}{L}\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ u_3 \end{array}$$

3b. Bar Element Example 2 : Self weight

$$\boxed{[K].\underline{d} = \underline{f}} \implies \frac{2EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a + f_1 \\ 2a \\ a \end{bmatrix} \quad \boxed{Eq\ 3}$$

$$a = \frac{AL\rho g}{4}$$

As in previous example, partition out 2nd and 3rd equations where forces are known

$$\frac{2EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2a \\ a \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{La}{2EA}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Matrix inversion (reminder)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(\mathrm{ad} - \mathrm{bc})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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3b. Bar Element Example 2 : Self weight

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{(2\times 1-(-1)^2)}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2-1}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

a

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{L}{2EA}\times\frac{AL\rho g}{4}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{L^2\rho g}{8E}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

First equation from $Eq\ 3$ then gives

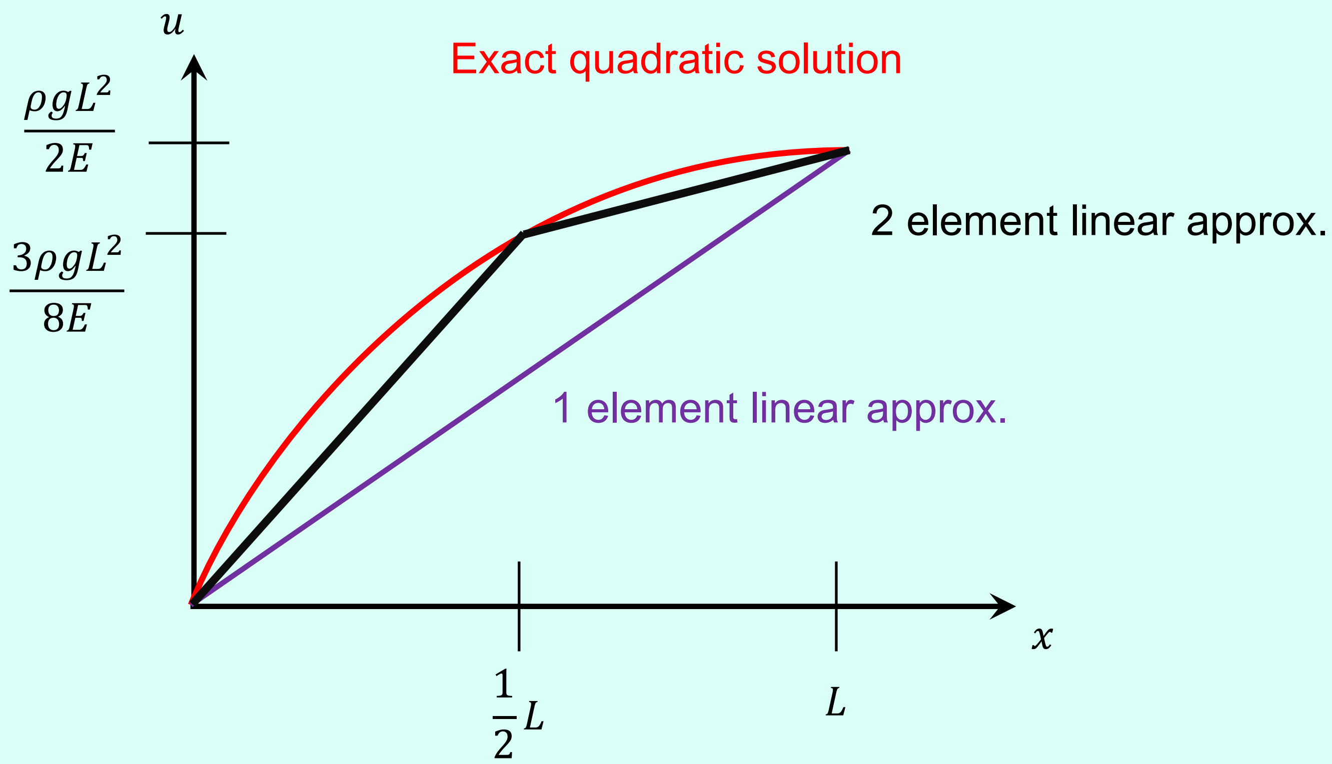
$$\frac{2EA}{L}[1.(0)-1.(u_2).0(u_3)] = a + f_1$$

$$\Rightarrow \quad a + f_1 = \frac{2EA}{L}(-u_2) = -\frac{2EA}{L}.\frac{3L^2\rho g}{8E} = -\frac{3AL\rho g}{4}$$

Reaction = weight of bar

$$\Rightarrow \quad f_1 = -\frac{3AL\rho g}{4} - \frac{AL\rho g}{4} = -AL\rho g\left(\frac{3}{4}+\frac{1}{4}\right) \quad \Rightarrow \boxed{f_1 = -AL\rho g}$$

3b. Bar Element Example 2 : Self weight



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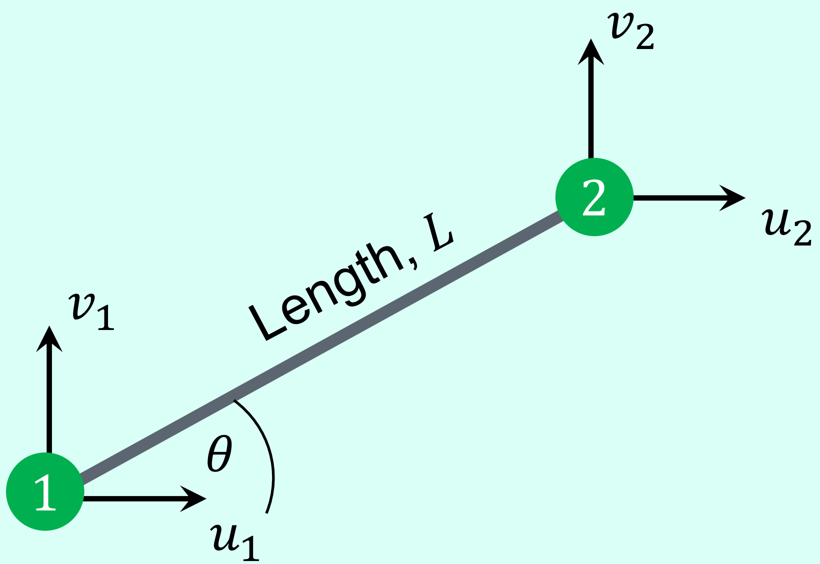
3c. Bar elements for 2D frameworks

Consider a bar element with orientation θ. In 2D the horizontal and vertical displacements are u and v.

DOF



Only displacements parallel to bar axis cause extension/compression